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GENERAL SOLUTIONS TO A CLASS OF UNSTEADY HEAT CONDUCTION PROBLEMS IN A RECTANGULAR PARALLELEPIPED

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Abstract—General expressions are presented for the three-dimensional and unsteady heating of a finite solid rectangular parallelepiped under the influence of an arbitrary volume heat source and an arbitrary initial temperature distribution when convective type of time-dependent boundary conditions are prescribed on the six plane surfaces. These expressions given in various forms and not available hitherto, contain the solution of numerous special problems of technological importance. Corresponding general expressions for the rectangle and for the slab are deduced as limiting cases. The particular problem treated recently by Cobble is shown to be a very special case of the results derived here for the parallelepiped.

NOMENCLATURE

A_{kmn},	coefficient defined by equation (16);
a,	half length of parallelepiped in x-direction;
B_i,	Biot numbers defined in equations (12);
b,	half length of parallelepiped in y-direction;
c,	half length of parallelepiped in z-direction;
C_k, D_m, E_n,	coefficients defined in equations (13);
$F(x, y, z)$,	initial temperature distribution in parallelepiped;
$f_1(y, z, t), f_2(y, z, t)$,	source functions at the surfaces $x = -a$, $x = a$, respectively;
$f_3(x, z, t), f_4(x, z, t)$,	source functions at the surfaces $y = -b$, $y = b$, respectively;
$f_5(x, y, t), f_6(x, y, t)$,	source functions at the surfaces $z = -c$, $z = c$, respectively;
f'_i,	surface source functions related to f_i by equation (8);
$H(x)$,	Heaviside unit function of x;
h_1, h_2,	heat-transfer coefficients on the surfaces $x = -a$, $x = a$, respectively;
h_3, h_4,	heat-transfer coefficients on the surfaces $y = -b$, $y = b$, respectively;
h_5, h_6,	heat-transfer coefficients on the surfaces $z = -c$, $z = c$, respectively;
i (subscript),	$1, 2, \ldots, 6$ (unless otherwise stated);
j (subscript),	$0, 1, 2, \ldots, 6$ (unless otherwise stated);
K,	thermal conductivity;
k,	constant (real, positive or negative);
k (subscript),	$0, 1, 2, \ldots, \infty$;
$L_{km}(z, z')$,	Green's function defined by equation (32);
M,	function defined by equation (40);
m (subscript),	$0, 1, 2, \ldots, \infty$;
N,	function defined by equation (48);
n (subscript),	$0, 1, 2, \ldots, \infty$;

B

P,	function defined by equation (55);
$Q(x, y, z, t)$,	internal heat source per unit time and per unit volume;
$Q'(x, y, z, t)$,	internal source related to Q by equation (9);
$T'(x, y, z, t)$,	unsteady temperature distribution defined by equations (1), (2), (3), and related to T by equation (4);
$T(x, y, z, t)$,	unsteady temperature distribution defined by equations (5), (6), (7), and given by expression (15), (20) or (21);
$T_{0j}(x, y, z, t)$,	pseudo-steady temperature distributions of order zero, defined by equations (18) and (19);
$T_{00}(x, y, z, t)$,	pseudo-steady temperature distribution defined by equations (26) and (27), and given by expression (33);
$T(x, y, t)$,	two-dimensional unsteady temperature distribution given by expression (60), (62) or (63);
$T_{00}(x, y, t)$,	two-dimensional pseudo-steady temperature distribution given by expression (65), (66) or (69);
$T(x, t)$,	one-dimensional unsteady temperature distribution given by expression (74), (75) or (76);
$T_{00}(x, t)$,	one dimensional pseudo-steady temperature distribution given by expression (77) or (78);
t,	time;
U, V, W,	defined in equations (25);
$U(x, y, z, t)$,	pseudo-steady temperature distribution defined by equations (34) and (35), and given by expression (41);
$U(x, y, t)$,	two-dimensional pseudo-steady temperature distribution given by expression (72);
$U(x, t)$,	one-dimensional pseudo-steady temperature distribution given by expression (79);
$V(x, y, z, t)$,	pseudo-steady temperature distribution defined by equations (42) and (43), and given by expression (49);
$V(x, y, t)$,	two-dimensional pseudo-steady temperature distribution given by expression (73);
$W(x, y, z)$,	pseudo-steady temperature distribution defined by equations (50) and (51), and given by expression (56);
x, y, z,	Cartesian coordinates;
$X_k(x)$, $Y_m(y)$, $Z_n(z)$,	eigenfunctions defined by equations (10);
$X_k(a)$, $Y_m(b)$, $Z_n(c)$,	defined in equations (14).

Greek symbols

α_k, β_m, γ_n,	eigenvalues defined by equations (11);
δ_{ij}, δ_{0j},	Kronecker delta;
ϵ_{kn},	defined by equation (46);
ζ_n,	defined by equation (58) or (59);
η_k,	defined by equation (67);
κ,	thermal diffusivity;
$\Lambda_k(y, y')$,	Green's function defined by equation (68);

λ_{kmn},	defined by equation (17);
λ_{km},	defined by equation (61);
μ_{km},	defined by equation (30);
ν_{mn},	defined by equation (38);
ξ_m,	defined by equation (70);
$\Omega_m(x, x')$,	Green's function defined by equation (71);
$\widetilde{(\ )}, (\ )^*\ \overline{(\ )}$	finite trigonometric transforms of ( ) defined by equations (22a), (23a), (24a), respectively.

INTRODUCTION

In a recent issue of this journal Cobble [1] studied a transient heat conduction problem in a two-dimensional rectangular region under the influence of an arbitrarily prescribed internal heat source and with arbitrary initial condition. The boundary conditions employed in [1] are assumed to be homogeneous and of convective type, with equal heat transfer coefficients on the four edges. In the equation of heat conduction, Cobble [1] included a heat sink term proportional to temperature and representing the heat loss by convection from the two lateral sides of the rectangle.

In this study we consider a more general and complete problem for a rectangular parallelepiped of side lengths $2a, 2b, 2c$ and present, in various forms, the general solution for the unsteady temperature distribution. From these, the solution of numerous special unsteady and steady heat flow problems in rectangular regions follow readily through appropriate specialization of the volume- and surface heat sources, and the boundary- and initial conditions employed in the present study. Some of these special problems have been solved by Carslaw and Jaeger [2]. The end result expressed by equation (45) of Cobble [1] is also readily deduced as another special case from one of the expressions established here for the parallelepiped.

PROBLEM STATEMENT

Consider a three-dimensional rectangular stationary region enclosed by the planes $x = \pm a$, $y = \pm b, z = \pm c$ in the Cartesian coordinate system Oxyz. The flow of heat by conduction is governed by the equation

$$\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} + \frac{\partial^2 T'}{\partial z^2} - kT' + \frac{1}{K}Q'(x, y, z, t) = \frac{1}{\kappa}\frac{\partial T'}{\partial t} \qquad (1)$$
$$(|x| < a, \quad |y| < b, \quad |z| < c, \quad t > 0)$$

where $T' = T'(x, y, z, t)$ is the unsteady temperature distribution and kT' represents a heat source proportional to temperature. Associated with (1) are the boundary conditions

$$(-)^i K\frac{\partial T'}{\partial x} + h_i T' = f_i'(y, z, t) \qquad (x = (-)^i a, \quad |y| < b, \quad |z| < c, \quad t > 0; \quad i = 1, 2) \qquad (2a)$$

$$(-)^i K\frac{\partial T'}{\partial y} + h_i T' = f_i'(x, z, t) \qquad (|x| < a, \quad y = (-)^i b, \quad |z| < c, \quad t > 0; i = 3, 4) \qquad (2b)$$

$$(-)^i K\frac{\partial T'}{\partial z} + h_i T' = f_i'(x, y, t) \qquad (|x| < a, \quad |y| < b, \quad z = (-)^i c, \quad t > 0; \quad i = 5, 6) \qquad (2c)$$

where $h_i \geqslant 0$ $(i = 1, 2, \ldots, 6)$ are the surface heat-transfer coefficients and $f_i'(i = 1, 2, \ldots, 6)$ are integrable functions prescribed on the six surfaces. The statement of the problem is completed by specifying the initial conditions as

$$T' = F(x, y, z) \qquad (|x| \leqslant a, \quad |y| \leqslant b, \quad |z| \leqslant c, \quad t = 0) \tag{3}$$

where $F(x, y, z)$ is a prescribed integrable function representing the initial temperature distribution in the parallelepiped.

The problem can be restated in a somewhat simpler form by use of the well-known substitution of

$$T'(x, y, z, t) = T(x, y, z, t)\,\mathrm{e}^{-k\kappa t} \tag{4}$$

whereby the system of equations (1), (2) and (3) becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{K} Q(x, y, z, t) = \frac{1}{\kappa}\frac{\partial T}{\partial t} \qquad (|x| < a, \quad |y| < b, \quad |z| < c, \quad t > 0) \tag{5}$$

$$(-)^i K \frac{\partial T}{\partial x} + h_i T = f_i(y, z, t) \qquad (x = (-)^i a, \quad |y| < b, \quad |z| < c, \quad t > 0; \quad i = 1, 2) \tag{6a}$$

$$(-)^i K \frac{\partial T}{\partial y} + h_i T = f_i(x, z, t) \qquad (|x| < a, \quad y = (-)^i b, |z| < c, \quad t > 0; \quad i = 3, 4) \tag{6b}$$

$$(-)^i K \frac{\partial T}{\partial z} + h_i T = f_i(x, y, t) \qquad (|x| < a, \quad |y| < b, \quad z = (-)^i c, \quad t > 0; \quad i = 5, 6) \tag{6c}$$

$$T = F(x, y, z) \qquad (|x| \leqslant a, \quad |y| \leqslant b, \quad |z| \leqslant c, \quad t = 0) \tag{7}$$

where

$$f_i = f'_i\,\mathrm{e}^{k\kappa t} \qquad (i = 1, 2, \ldots, 6) \tag{8}$$

$$Q = Q'\,\mathrm{e}^{k\kappa t} \tag{9}$$

SOLUTION

For the solution of $T(x, y, z, t)$ from the system of equations (5), (6) and (7), we define the following auxiliary functions

$$X_k(x) = \cos \alpha_k \left(1 + \frac{x}{a}\right) + \frac{B_1}{\alpha_k} \sin \alpha_k \left(1 + \frac{x}{a}\right) \qquad (k = 0, 1, 2, \ldots) \tag{10a}$$

$$Y_m(y) = \cos \beta_m \left(1 + \frac{y}{b}\right) + \frac{B_3}{\beta_m} \sin \beta_m \left(1 + \frac{y}{b}\right) \qquad (m = 0, 1, 2, \ldots) \tag{10b}$$

$$Z_n(z) = \cos \gamma_n \left(1 + \frac{z}{c}\right) + \frac{B_5}{\gamma_n} \sin \gamma_n \left(1 + \frac{z}{c}\right) \qquad (n = 0, 1, 2, \ldots) \tag{10c}$$

where α_k, β_m and γ_n are the kth, the mth and the nth non-negative roots of

$$(B_1 + B_2)\,\alpha_k \cos 2\alpha_k = (\alpha_k^2 - B_1 B_2) \sin 2\alpha_k \tag{11a}$$

$$(B_3 + B_4)\,\beta_m \cos 2\beta_m = (\beta_m^2 - B_3 B_4) \sin 2\beta_m \tag{11b}$$

$$(B_5 + B_6)\,\gamma_n \cos 2\gamma_n = (\gamma_n^2 - B_5 B_6) \sin 2\gamma_n \tag{11c}$$

respectively, and the coefficients B_i $(i = 1, 2, \ldots, 6)$ are defined as

$$B_1 = \frac{h_1 a}{K} \qquad B_2 = \frac{h_2 a}{K} \tag{12a}$$

$$B_3 = \frac{h_3 b}{K} \qquad B_4 = \frac{h_4 b}{K} \tag{12b}$$

$$B_5 = \frac{h_5 c}{K} \qquad B_6 = \frac{h_6 c}{K}\ . \tag{12c}$$

It follows from equations (10) and (11) that

$$\left.\begin{aligned}\frac{1}{C_k} = \int_{-a}^{a} X_k^2(x)\,\mathrm{d}x &= \frac{a(\alpha_k^2 + B_1^2)(\alpha_k^2 + B_2^2) + (a/2)(B_1 + B_2)(\alpha_k^2 + B_1 B_2)}{\alpha_k^2(\alpha_k^2 + B_2^2)}, \\ &= 2a, \qquad\qquad \alpha_0 = 0\end{aligned}\right\} \tag{13a}$$

$$\left.\begin{aligned}\frac{1}{D_m} = \int_{-b}^{b} Y_m^2(y)\,\mathrm{d}y &= \frac{b(\beta_m^2 + B_3^2)(\beta_m^2 + B_4^2) + (b/2)(B_3 + B_4)(\beta_m^2 + B_3 B_4)}{\beta_m^2(\beta_m^2 + B_4^2)}, \\ &= 2b, \qquad\qquad \beta_0 = 0\end{aligned}\right\} \tag{13b}$$

$$\left.\begin{aligned}\frac{1}{E_n} = \int_{-c}^{c} Z_n^2(z)\,\mathrm{d}z &= \frac{c(\gamma_n^2 + B_5^2)(\gamma_n^2 + B_6^2) + (c/2)(B_5 + B_6)(\gamma_n^2 + B_5 B_6)}{\gamma_n^2(\gamma_n^2 + B_6^2)}, \\ &= 2c, \qquad\qquad \gamma_0 = 0\end{aligned}\right\} \tag{13c}$$

and

$$X_k(-a) = 1, \qquad X_k(a) = \left(\frac{\alpha_k^2 + B_1^2}{\alpha_k^2 - B_1 B_2}\right)\cos 2\alpha_k \tag{14a}$$

$$Y_m(-b) = 1, \qquad Y_m(b) = \left(\frac{\beta_m^2 + B_3^2}{\beta_m^2 - B_3 B_4}\right)\cos 2\beta_m \tag{14b}$$

$$Z_n(-c) = 1, \qquad Z_n(c) = \left(\frac{\gamma_n^2 + B_5^2}{\gamma_n^2 - B_5 B_6}\right)\cos 2\gamma_n. \tag{14c}$$

The solution to the system of equations (5), (6) and (7) can now be written down directly from the general expression (16) given in [3]. The result is

$$\left.\begin{aligned}T(x, y, z, t) = \sum_{j=0}^{6} T_{0j}(x, y, z, t) + \sum_{k}^{\infty}\sum_{m}^{\infty}\sum_{n}^{\infty}{}_{(k=m=n\neq 0)} A_{kmn} X_k(x)\, Y_m(y)\, Z_n(z)\, \mathrm{e}^{-\lambda_{kmn}^2 \kappa t} \\ \left\{ \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} X_k(x)\, Y_m(y)\, Z_n(z)\left[F(x, y, z) - \sum_{j=0}^{6} T_{0j}(x, y, z, 0)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \right. \\ \left. - \sum_{j=0}^{6}\int_{0}^{t} \mathrm{e}^{\lambda_{kmn}^2 \kappa\tau}\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} X_k(x)\, Y_m(y)\, Z_n(z)\frac{\partial T_{0j}(x, y, z, \tau)}{\partial \tau}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\tau \right\}\end{aligned}\right\} \tag{15}$$

where

$$A_{kmn} = C_k D_m E_n \tag{16}$$

$$\lambda_{kmn}^2 = \left(\frac{\alpha_k}{a}\right)^2 + \left(\frac{\beta_m}{b}\right)^2 + \left(\frac{\gamma_n}{c}\right)^2 \tag{17}$$

and the $T_{0j}(x, y, z, t)$ functions are defined by the system of

$$\frac{\partial^2 T_{0j}}{\partial x^2} + \frac{\partial^2 T_{0j}}{\partial y^2} + \frac{\partial^2 T_{0j}}{\partial z^2} + \frac{\delta_{0j}}{K} Q(x, y, z, t) = 0 \qquad (|x| < a, \quad |y| < b, \quad |z| < c; \ j = 0, 1, 2, \ldots, 6) \tag{18}$$

$$\left.\begin{aligned} &(-)^i K \frac{\partial T_{0j}}{\partial x} + h_i T_{0j} = \delta_{ij} f_i(y, z, t) \qquad (x = (-)^i a, \quad |y| < b, \quad |z| < c; \quad i = 1, 2) && \text{(19a)} \\ &(-)^i K \frac{\partial T_{0j}}{\partial y} + h_i T_{0j} = \delta_{ij} f_i(x, z, t) \qquad (|x| < a, \quad y = (-)^i b, \quad |z| < c; \quad i = 3, 4) && \text{(19b)} \\ &(-)^i K \frac{\partial T_{0j}}{\partial z} + h_i T_{0j} = \delta_{ij} f_i(x, y, t) \qquad (|x| < a, \quad |y| < b, \quad z = (-)^i c; \quad i = 5, 6) && \text{(19c)} \end{aligned}\right\} (j = 0, 1, 2, \ldots, 6).$$

An expression alternative to (15) and in which the source functions $f_i (i = 1, 2, \ldots, 6)$ and Q appear more explicitly follows from the general expression (17) of [3] as

$$\begin{aligned} T(x, y, z, t) = \sum_{j=0}^{6} T_{0j}(x, y, z, t) + \sum_{k}^{\infty} \sum_{m}^{\infty} \sum_{n}^{\infty}{}_{(k=m=n\neq 0)} A_{kmn}\, X_k(x)\, Y_m(y)\, Z_n(z)\, \mathrm{e}^{-\lambda_{kmn}^2 \kappa t} \,. \\ \Bigg\{ \int_{-a}^{a} \int_{-b}^{b} \int_{-c}^{c} X_k(x)\, Y_m(y)\, Z_n(z) \left[F(x, y, z) - \frac{Q(x, y, z, 0)}{K\lambda_{kmn}^2} \right] \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ - \frac{1}{K\lambda_{kmn}^2} \Bigg[\int_{-b}^{b} \int_{-c}^{c} Y_m(y)\, Z_n(z) \left\{ f_1(y, z, 0) + X_k(a) f_2(y, z, 0) \right\} \mathrm{d}y\, \mathrm{d}z \\ + \int_{-a}^{a} \int_{-c}^{c} X_k(x)\, Z_n(z) \left\{ f_3(x, z, 0) + Y_m(b) f_4(x, z, 0) \right\} \mathrm{d}x\, \mathrm{d}z \\ + \int_{-a}^{a} \int_{-b}^{b} X_k(x)\, Y_m(y) \left\{ f_5(x, y, 0) + Z_n(c) f_6(x, y, 0) \right\} \mathrm{d}x\, \mathrm{d}y \Bigg] \\ - \frac{1}{K\lambda_{kmn}^2} \int_{0}^{t} \mathrm{e}^{\lambda_{kmn}^2 \kappa \tau} \frac{\partial}{\partial \tau} \Bigg[\int_{-a}^{a} \int_{-b}^{b} \int_{-c}^{c} X_k(x)\, Y_m(y)\, Z_n(z)\, Q(x, y, z, \tau)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ + \int_{-b}^{b} \int_{-c}^{c} Y_m(y)\, Z_n(z) \left\{ f_1(y, z, \tau) + X_k(a) f_2(y, z, \tau) \right\} \mathrm{d}y\, \mathrm{d}z \\ + \int_{-a}^{a} \int_{-c}^{c} X_k(x)\, Z_n(z) \left\{ f_3(x, z, \tau) + Y_m(b) f_4(x, z, \tau) \right\} \mathrm{d}x\, \mathrm{d}z \end{aligned}$$

$$+\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\,\{f_5(x,y,\tau)+Z_n(c)f_6(x,y,\tau)\}\,\mathrm{d}x\,\mathrm{d}y\Bigg]\mathrm{d}\tau\Bigg\}\,. \tag{20}$$

It is to be noted that the expressions (15) and (20) are not valid in the event of $h_1 = h_2 = h_3 = h_4 = h_5 = h_6 = 0$. The corresponding expressions for the parallelepiped in this particular case can be readily written down by use of the general results given by equations (22a) and (22b) in [4].

Another expression alternative to (15) follows from [3] as

$$T(x,y,z,t) = \sum^{\infty}\sum^{\infty}\sum^{\infty} A_{kmn}\,X_k(x)\,Y_m(y)\,Z_n(z)\,\mathrm{e}^{-\lambda_{kmn}^2 \kappa t}\,.$$

$$\cdot\Bigg\{\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} X_k(x)\,Y_m(y)\,Z_n(z)\,F(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$+\frac{\kappa}{K}\int_{0}^{t}\mathrm{e}^{\lambda_{kmn}^2\kappa\tau}\Bigg[\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} X_k(x)\,Y_m(y)\,Z_n(z)\,Q(x,y,z,\tau)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$+\int_{-b}^{b}\int_{-c}^{c} Y_m(y)\,Z_n(z)\,\{f_1(y,z,\tau)+X_k(a)f_2(y,z,\tau)\}\,\mathrm{d}y\,\mathrm{d}z$$

$$+\int_{-a}^{a}\int_{-c}^{c} X_k(x)\,Z_n(z)\,\{f_3(x,z,\tau)+Y_m(b)f_4(x,z,\tau)\}\,\mathrm{d}x\,\mathrm{d}z$$

$$+\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\,\{f_5(x,y,\tau)+Z_n(c)f_6(x,y,\tau)\}\,\mathrm{d}x\,\mathrm{d}y\Bigg]\mathrm{d}\tau,\Bigg\}\,. \tag{21}$$

While the $T_{0j}(x,y,z,t)$ functions no longer appear in this expression, the usefulness of (21) is limited to the particular case in which the time-dependent source functions $f_i (i = 1, 2, \ldots, 6)$ and Q are instantaneous pulses at zero time. The convergence of (21) is not uniform unless $f_i \equiv 0 (i = 1, 2, \ldots, 6)$. On the other hand, expressions (15) and (20) converge uniformly and at a faster rate than (21) and are especially well suited in the event that f_i and Q are continuous pulses released at zero time. Some special cases of (21) can be found in [5].

SOLUTIONS FOR $T_{0j}(x, y, z, t)$

The so called pseudo-steady functions of order zero, $T_{0j}(x,y,z,t)$ $(j = 0, 1, 2, \ldots, 6)$, appearing in (15) and (20) are yet to be determined. To this end, we first define three finite transforms as follows:

(a) Finite trigonometric transform with respect to x:

$$\tilde{T}_{0j}(k,y,z,t) = \int_{-a}^{a} T_{0j}(x,y,z,t)\,X_k(x)\,\mathrm{d}x \tag{22a}$$

the inverse transform being

$$T_{0j}(x, y, z, t) = \sum_{k}^{\infty} C_k \, X_k(x) \, \tilde{T}_{0j}(k, y, z, t) \tag{22b}$$

where C_k and $X_k(x)$ are given by (13a) and (10a), respectively, and the summation is taken over the non-negative roots of (11a).

(b) Finite trigonometric transform with respect to y:

$$T_{0j}^*(x, m, z, t) = \int_{-b}^{b} T_{0j}(x, y, z, t) \, Y_m(y) \, \mathrm{d}y \tag{23a}$$

the inverse transform being

$$T_{0j}(x, y, z, t) = \sum_{m}^{\infty} D_m \, Y_m(y) \, T_{0j}^*(x, m, z, t) \tag{23b}$$

where D_m and $Y_m(y)$ are given by (13b) and (10b), respectively, and the summation is taken over the non-negative roots of (11b).

(c) Finite trigonometric transform with respect to z:

$$\bar{T}_{0j}(x, y, n, t) = \int_{-c}^{c} T_{0j}(x, y, z, t) \, Z_n(z) \, \mathrm{d}z \tag{24a}$$

the inverse transform being

$$T_{0j}(x, y, z, t) = \sum_{n}^{\infty} E_n \, Z_n(z) \, \bar{T}_{0j}(x, y, n, t) \tag{24b}$$

where E_n and $Z_n(z)$ are given by (13c) and (10c), respectively, and the summation is taken over the non-negative roots of (11c). These three sets of transform pairs enable the determination of the $T_{0j}(x, y, z, t)$ functions in a particularly concise manner. For convenience let

$$U(x, y, z, t) = T_{01}(x, y, z, t) + T_{02}(x, y, z, t) \tag{25a}$$

$$V(x, y, z, t) = T_{03}(x, y, z, t) + T_{04}(x, y, z, t) \tag{25b}$$

$$W(x, y, z, t) = T_{05}(x, y, z, t) + T_{06}(x, y, z, t). \tag{25c}$$

Determination of $T_{00}(x, y, z, t)$

From (18) and (19), with $j = 0$, the differential equation and the boundary conditions defining $T_{00}(x, y, z, t)$ are

$$\frac{\partial^2 T_{00}}{\partial x^2} + \frac{\partial^2 T_{00}}{\partial y^2} + \frac{\partial^2 T_{00}}{\partial z^2} + \frac{1}{K} Q(x, y, z, t) = 0 \qquad (|x| < a, \ |y| < b, \ |z| < c) \tag{26}$$

$$(-)^i K \frac{\partial T_{00}}{\partial x} + h_i T_{00} = 0 \qquad (x = (-)^i a, \ |y| < b, \ |z| < c; \ i = 1, 2) \tag{27a}$$

$$(-)^i K \frac{\partial T_{00}}{\partial y} + h_i T_{00} = 0 \qquad (|x| < a, \ y = (-)^i b, \ |z| < c; \ i = 3, 4) \tag{27b}$$

$$(-)^i K \frac{\partial T_{00}}{\partial z} + h_i T_{00} = 0 \qquad (|x| < a, \ |y| < b, \ z = (-)^i c; \ i = 5, 6). \tag{27c}$$

The transformation of the system of (26) and (27) first by (22a) and then by (23a) results in

$$\frac{\partial^2 \tilde{T}^*_{00}(k,m,z,t)}{\partial z^2} - \frac{\mu^2_{km}}{c^2}\tilde{T}^*_{00}(k,m,z,t) + \frac{1}{K}\tilde{Q}^*(k,m,z,t) = 0 \qquad (|z| < c) \tag{28}$$

$$(-)^i K \frac{\partial \tilde{T}^*_{00}}{\partial z} + h_i \tilde{T}^*_{00} = 0 \qquad (z = (-)^i c;\quad i = 5, 6) \tag{29}$$

where

$$\frac{\mu^2_{km}}{c^2} = \frac{\alpha^2_k}{a^2} + \frac{\beta^2_m}{b^2}. \tag{30}$$

Under the restriction that h_1, h_2, h_3, h_4, h_5 and h_6 are not to be simultaneously zero, the solution to the system of (28) and (29) is obtained as

$$\tilde{T}^*_{00}(k,m,z,t) = \frac{c}{K\mu_{km}}\left[\int_0^z \sinh \mu_{km}\left(\frac{z'}{c} - \frac{z}{c}\right)\tilde{Q}^*(k,m,z',t)\,dz' + \int_{-c}^{c} \{H(z')\,L_{km}(z',z) + H(-z')\,L_{km}(z,z')\}\,\tilde{Q}^*(k,m,z',t)\,dz'\right] \tag{31}$$

where

$$L_{km}(z,z') = \frac{\left\{\mu_{km}\cosh\mu_{km}\left(1-\frac{z}{c}\right) + B_6\sinh\mu_{km}\left(1-\frac{z}{c}\right)\right\}\left\{\mu_{km}\cosh\mu_{km}\left(1+\frac{z'}{c}\right) + B_5\sinh\mu_{km}\left(1+\frac{z'}{c}\right)\right\}}{(\mu^2_{km} + B_5B_6)\sinh 2\mu_{km} + \mu_{km}(B_5 + B_6)\cosh 2\mu_{km}} \tag{32}$$

and $H(z)$ is the Heaviside unit function. The successive inversion of (31) by (23b) and (22b) gives the result

$$T_{00}(x,y,z,t) = \sum_k^\infty \sum_m^\infty \left(\frac{c}{K}\right)\frac{C_k D_m}{\mu_{km}} X_k(x)\,Y_m(y) \int_{-a}^{a}\int_{-b}^{b}\left[\int_0^z \sinh\mu_{km}\left(\frac{z'}{c} - \frac{z}{c}\right)Q(x,y,z',t)\,dz' + \int_{-c}^{c}\{H(z')\,L_{km}(z',z) + H(-z')\,L_{km}(z,z')\}\,Q(x,y,z',t)\,dz'\right]X_k(x)\,Y_m(y)\,dx\,dy. \tag{33}$$

It is to be noted that T_{00} can be obtained in two alternative forms by the repeated use of the transformations (22a) and (24a), or by (23a) and (24a). A practical choice from among the three forms of the solutions for $T_{00}(x, y, z, t)$ should be the one involving the simplest two of the three frequency equations (11a), (11b) and (11c).

Determination of $U(x, y, z, t)$

From (18) and (19) with $j = 1, 2$, and from (25a) it follows that the differential equation and the boundary conditions for $U(x, y, z, t)$ are

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \qquad (|x| < a, \quad |y| < b, \quad |z| < c) \tag{34}$$

$$(-)^i K \frac{\partial U}{\partial x} + h_i U = f_i(y, z, t) \qquad (x = (-)^i a, \quad |y| < b, \quad |z| < c; \quad i = 1, 2) \tag{35a}$$

$$(-)^i K \frac{\partial U}{\partial y} + h_i U = 0 \qquad (|x| < a, \quad y = (-)^i b, \quad |z| < c; \quad i = 3, 4) \tag{35b}$$

$$(-)^i K \frac{\partial U}{\partial z} + h_i U = 0 \qquad (|x| < a, \quad |y| < b, \quad z = (-)^i c; \quad i = 5, 6). \tag{35c}$$

The system of (34) and (35) is now transformed successively by (23a) and (24a) to give

$$\left(\frac{\partial^2}{\partial x^2} - \frac{v_{mn}^2}{a^2}\right) \bar{U}^*(x, m, n, t) = 0 \qquad (|x| < a) \tag{36}$$

$$(-)^i K \frac{\partial \bar{U}^*}{\partial x} + h_i \bar{U}^* = \bar{f}_i^*(m, n, t) \qquad (x = (-)^i a; \quad i = 1, 2) \tag{37}$$

where

$$\frac{v_{mn}^2}{a^2} = \frac{\beta_m^2}{b^2} + \frac{\gamma_n^2}{c^2}. \tag{38}$$

Under the restriction that h_1, h_2, h_3, h_4, h_5 and h_6 are not to be simultaneously zero, the solution to the system of (36) and (37) is obtained as

$$\bar{U}^*(x, m, n, t) = \frac{a}{KM(v_{mn})} \left[\left\{ v_{mn} \cosh v_{mn} \left(1 - \frac{x}{a}\right) + B_2 \sinh v_{mn} \left(1 - \frac{x}{a}\right) \right\} \bar{f}_1^*(m, n, t) \right.$$
$$\left. + \left\{ v_{mn} \cosh v_{mn} \left(1 + \frac{x}{a}\right) + B_1 \sinh v_{mn} \left(1 + \frac{x}{a}\right) \right\} \bar{f}_2^*(m, n, t) \right] \tag{39}$$

where

$$M(v_{mn}) = (v_{mn}^2 + B_1 B_2) \sinh 2v_{mn} + v_{mn}(B_1 + B_2) \cosh 2v_{mn}. \tag{40}$$

Inverting (39) by (24b) and (23b) in succession we have

$$U(x, y, z, t) = \sum_m^\infty \sum_n^\infty \left(\frac{a}{K}\right) \frac{D_m E_n}{M(v_{mn})} Y_m(y) Z_n(z) \cdot$$
$$\left[\left\{ v_{mn} \cosh v_{mn} \left(1 - \frac{x}{a}\right) + B_2 \sinh v_{mn} \left(1 - \frac{x}{a}\right) \right\} \int_{-b}^{b} \int_{-c}^{c} f_1(y, z, t) Y_m(y) Z_n(z) \, dy \, dz \right.$$
$$\left. + \left\{ v_{mn} \cosh v_{mn} \left(1 + \frac{x}{a}\right) + B_1 \sinh v_{mn} \left(1 + \frac{x}{a}\right) \right\} \int_{-b}^{b} \int_{-c}^{c} f_2(y, z, t) Y_m(y) Z_n(z) \, dy \, dz \right]. \tag{41}$$

With $f_2 = 0$, U becomes $T_{01}(x, y, z, t)$ and with $f_1 = 0$, U becomes $T_{02}(x, y, z, t)$.

From (18) and (19) with $j = 3, 4$, and from (25b) it follows that the differential equation and the boundary conditions for $V(x, y, z, t)$ are

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (|x| < a, \quad |y| < b, \quad |z| < c) \tag{42}$$

$$(-)^i K \frac{\partial V}{\partial x} + h_i V = 0 \qquad (x = (-)^i a, \quad |y| < b, \quad |z| < c; \quad i = 1, 2) \tag{43a}$$

$$(-)^i K \frac{\partial V}{\partial y} + h_i V = f_i(x, z, t) \qquad (|x| < a, \quad y = (-)^i b, \quad |z| < c; \quad i = 3, 4) \tag{43b}$$

$$(-)^i K \frac{\partial V}{\partial z} + h_i V = 0 \qquad (|x| < a, \quad |y| < b, \quad z = (-)^i c; \quad i = 5, 6). \tag{43c}$$

Transforming the system of (42) and (43) by (22a) and (24a) we have

$$\left(\frac{\partial^2}{\partial y^2} - \frac{\epsilon_{kn}^2}{b^2}\right) \bar{\bar{V}}(k, y, n, t) = 0 \qquad (|y| < b) \tag{44}$$

$$(-)^i K \frac{\partial \bar{\bar{V}}}{\partial y} + h_i \bar{\bar{V}} = \bar{\bar{f}}_i \qquad (y = (-)^i b; \quad i = 3, 4) \tag{45}$$

where

$$\frac{\epsilon_{kn}^2}{b^2} = \frac{\alpha_k^2}{a^2} + \frac{\gamma_n^2}{c^2}. \tag{46}$$

Under the restriction that h_1, h_2, h_3, h_4, h_5 and h_6 are not to be simultaneously zero, the solution to the system of (44) and (45) is obtained as

$$\bar{\bar{V}}(k, y, n, t) = \frac{b}{KN(\epsilon_{kn})} \left[\left\{ \epsilon_{kn} \cosh \epsilon_{kn} \left(1 - \frac{y}{b}\right) + B_4 \sinh \epsilon_{kn} \left(1 - \frac{y}{b}\right) \right\} \bar{\bar{f}}_3(k, n, t) \right.$$
$$\left. + \left\{ \epsilon_{kn} \cosh \epsilon_{kn} \left(1 + \frac{y}{b}\right) + B_3 \sinh \epsilon_{kn} \left(1 + \frac{y}{b}\right) \right\} \bar{\bar{f}}_4(k, n, t) \right] \tag{47}$$

where

$$N(\epsilon_{kn}) = (\epsilon_{kn}^2 + B_3 B_4) \sinh 2\epsilon_{kn} + \epsilon_{kn}(B_3 + B_4) \cosh 2\epsilon_{kn}. \tag{48}$$

Inverting (47) by (24b) and (22b) successively we get

$$V(x, y, z, t) = \sum_k^\infty \sum_n^\infty \left(\frac{b}{K}\right) \frac{C_k E_n}{N(\epsilon_{kn})} X_k(x)\, Z_n(z) \cdot$$
$$\cdot \left[\left\{ \epsilon_{kn} \cosh \epsilon_{kn} \left(1 - \frac{y}{b}\right) + B_4 \sinh \epsilon_{kn} \left(1 - \frac{y}{b}\right) \right\} \int_{-a}^{a} \int_{-c}^{c} f_3(x, z, t)\, X_k(x)\, Z_n(z)\, \mathrm{d}x\, \mathrm{d}z \right.$$
$$\left. + \left\{ \epsilon_{kn} \cosh \epsilon_{kn} \left(1 + \frac{y}{b}\right) + B_3 \sinh \epsilon_{kn} \left(1 + \frac{y}{b}\right) \right\} \int_{-a}^{a} \int_{-c}^{c} f_4(x, z, t)\, X_k(x) Z_n(z)\, \mathrm{d}x\, \mathrm{d}z \right]. \tag{49}$$

With $f_4 = 0$, V becomes $T_{03}(x, y, z, t)$ and with $f_3 = 0$, V becomes $T_{04}(x, y, z, t)$.

Determination of W(x, y, z, t)

From (18) and (19) with $j = 5, 6$, and from (25c) it follows that the differential equation and the boundary conditions for $W(x, y, z, t)$ are

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0 \qquad (|x| < a, \quad |y| < b, \quad |z| < c) \tag{50}$$

$$(-)^i K \frac{\partial W}{\partial x} + h_i W = 0 \qquad (x = (-)^i a, \quad |y| < b, \quad |z| < c; \quad i = 1, 2) \tag{51a}$$

$$(-)^i K \frac{\partial W}{\partial y} + h_i W = 0 \qquad (|x| < a, \quad y = (-)^i b, \quad |z| < c; \quad i = 3, 4) \tag{51b}$$

$$(-)^i K \frac{\partial W}{\partial z} + h_i W = f_i(x, y, t) \qquad (|x| < a, \quad |y| < b, \quad z = (-)^i c; \quad i = 5, 6) \tag{51c}$$

Transforming the system of (50) and (51) first by (22a) and then by (23a) we obtain

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\mu_{km}^2}{c^2}\right) \tilde{W}^*(k, m, z, t) = 0 \qquad (|z| < c) \tag{52}$$

$$(-)^i K \frac{\partial \tilde{W}^*}{\partial z} + h_i \tilde{W}^* = \tilde{f}_i^*(k, m, t) \qquad (z = (-)^i c; \quad i = 5, 6) \tag{53}$$

where μ_{km} is defined in (30). Under the restriction that h_1, h_2, h_3, h_4, h_5 and h_6 are not to be simultaneously zero, the solution to the system of (52) and (53) is obtained as

$$\begin{aligned} \tilde{W}^*(k, m, z, t) = \frac{c}{KP(\mu_{km})} \Bigg[& \left\{ \mu_{km} \cosh \mu_{km} \left(1 - \frac{z}{c}\right) + B_6 \sinh \mu_{km} \left(1 - \frac{z}{c}\right) \right\} \tilde{f}_5^*(k, m, t) \\ & + \left\{ \mu_{km} \cosh \mu_{km} \left(1 + \frac{z}{c}\right) + B_5 \sinh \mu_{km} \left(1 + \frac{z}{c}\right) \right\} \tilde{f}_6^*(k, m, t) \Bigg] \end{aligned} \tag{54}$$

where

$$P(\mu_{km}) = (\mu_{km}^2 + B_5 B_6) \sinh 2\mu_{km} + \mu_{km}(B_5 + B_6) \cosh 2\mu_{km}. \tag{55}$$

Inverting (54) by (23b) and (22b) in succession we have

$$\begin{aligned} W(x, y, z, t) = & \sum_k^\infty \sum_m^\infty \left(\frac{c}{K}\right) \frac{C_k D_m}{P(\mu_{km})} X_k(x)\, Y_m(y) \\ & \cdot \Bigg[\left\{ \mu_{km} \cosh \mu_{km} \left(1 - \frac{z}{c}\right) + B_6 \sinh \mu_{km} \left(1 - \frac{z}{c}\right) \right\} \int_{-a}^{a} \int_{-b}^{b} f_5(x, y, t)\, X_k(x)\, Y_m(y)\, \mathrm{d}x\, \mathrm{d}y \\ & + \left\{ \mu_{km} \cosh \mu_{km} \left(1 + \frac{z}{c}\right) + B_5 \sinh \mu_{km} \left(1 + \frac{z}{c}\right) \right\} \int_{-a}^{a} \int_{-b}^{b} f_6(x, y, t)\, X_k(x)\, Y_m(y)\, \mathrm{d}x\, \mathrm{d}y \Bigg]. \end{aligned} \tag{56}$$

With $f_6 = 0$, W becomes $T_{05}(x, y, z, t)$ and with $f_5 = 0$, W becomes $T_{06}(x, y, z, t)$. This completes the determination of the $T_{0j}(j = 0, 1, 2, \ldots, 6)$ functions.

PARTICULAR CASES

(1) *Two-dimensional problems*

An important special case arises when the functions f_1, f_2, f_3, f_4, F and Q are independent of z and, in addition, the faces $z = \pm c$ are insulated, i.e. $h_5 = h_6 = f_5 = f_6 = 0$. This is the case of the two-dimensional unsteady heat flow in the rectangle with sides $2a$ and $2b$. To obtain the corresponding general expressions for this case we note that, in the event of $h_5 = h_6 = 0$, the frequency equation (11c) reduces to

$$\sin 2\gamma_n = 0 \tag{57}$$

and from

$$\zeta_n = \int_{-c}^{c} Z_n(z)\,\mathrm{d}z = \frac{2cZ_n(0)\sin\gamma_n}{\gamma_n} \tag{58}$$

we have

$$\zeta_n = c\frac{\sin 2\gamma_n}{\gamma_n}. \tag{59}$$

From (57) and (59) it follows that $\zeta_n = 0$ for all n except for $n = 0$, corresponding to $\gamma_0 = 0$ which is also a root of (57). For $\gamma_0 = 0$ we obtain $Z_0(z) = 1$, $\zeta_0 = 2c$ and $E_0^{-1} = 2c$. Furthermore, since $f_i(i = 1, 2, 3, 4)$, F and Q are now independent of z, only the term corresponding to $n = 0$ contributes to the n-summation in (15) which becomes

$$T(x, y, t) = \sum_{j=0}^{4} T_{0j}(x, y, t) + \sum_{\substack{k\ m \\ (k=m\neq 0)}}^{\infty\ \infty}\sum C_k D_m X_k(x)\,Y_m(y)\,\mathrm{e}^{-\lambda_{km}^2\kappa t}$$
$$\cdot\left\{\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\left[F(x, y) - \sum_{j=0}^{4} T_{0j}(x, y, 0)\right]\mathrm{d}x\,\mathrm{d}y\right.$$
$$\left. - \sum_{j=0}^{4}\int_{0}^{t}\mathrm{e}^{\lambda_{km}^2\kappa\tau}\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\frac{\partial T_{0j}(x, y, \tau)}{\partial\tau}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\right\} \tag{60}$$

where

$$\lambda_{km}^2 = \left(\frac{\alpha_k}{a}\right)^2 + \left(\frac{\beta_m}{b}\right)^2. \tag{61}$$

Similarly, expressions (20) and (21) reduce to

$$T(x, y, t) = \sum_{j=0}^{4} T_{0j}(x, y, t) + \sum_{\substack{k\ m \\ (k=m\neq 0)}}^{\infty\ \infty}\sum C_k D_m X_k(x)\,Y_m(y)\,\mathrm{e}^{-\lambda_{km}^2\kappa t}$$
$$\cdot\left\{\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\left[F(x, y) - \frac{Q(x, y, 0)}{K\lambda_{km}^2}\right]\mathrm{d}x\,\mathrm{d}y\right.$$

$$
-\frac{1}{K\lambda_{km}^2}\left[\int_{-b}^{b} Y_m(y)\{f_1(y,0)+X_k(a)\,f_2(y,0)\}\,\mathrm{d}y+\int_{-a}^{a} X_k(x)\{f_3(x,0)+Y_m(b)\,f_4(x,0)\}\,\mathrm{d}x\right]
$$

$$
-\frac{1}{K\lambda_{km}^2}\int_0^t \mathrm{e}^{\lambda_{km}^2\kappa\tau}\frac{\partial}{\partial\tau}\left[\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\,Q(x,y,\tau)\,\mathrm{d}x\,\mathrm{d}y\right.
$$

$$
\left.+\int_{-b}^{b} Y_m(y)\{f_1(y,\tau)+X_k(a)\,f_2(y,\tau)\}\,\mathrm{d}y+\int_{-a}^{a} X_k(x)\{f_3(x,\tau)+Y_m(b)\,f_4(x,\tau)\}\,\mathrm{d}x\right]\mathrm{d}\tau\Bigg\} \tag{62}
$$

and

$$
T(x,y,t)=\sum_k^{\infty}\sum_m^{\infty} C_kD_mX_k(x)\,Y_m(y)\,\mathrm{e}^{-\lambda_{km}^2\kappa t}\left\{\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\,F(x,y)\,\mathrm{d}x\,\mathrm{d}y\right.
$$

$$
+\frac{\kappa}{K}\int_0^t \mathrm{e}^{\lambda_{km}^2\kappa\tau}\left[\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\,Q(x,y,\tau)\,\mathrm{d}x\,\mathrm{d}y\right.
$$

$$
\left.\left.+\int_{-b}^{b} Y_m(y)\{f_1(y,\tau)+X_k(a)\,f_2(y,\tau)\}\,\mathrm{d}y+\int_{-a}^{a} X_k(x)\{f_3(x,\tau)+Y_m(b)\,f_4(x,\tau)\}\,\mathrm{d}x\right]\mathrm{d}\tau\right\} \tag{63}
$$

respectively. Since now

$$
\int_0^z \sinh\mu_{km}\left(\frac{z'}{c}-\frac{z}{c}\right)\mathrm{d}z'+\int_{-c}^{c}\{H(z')\,L_{km}(z',z)+H(-z')\,L_{km}(z,z')\}\,\mathrm{d}z'=\frac{c}{\mu_{km}} \tag{64}
$$

expression (33) becomes

$$
T_{00}(x,y,t)=\sum_k^{\infty}\sum_m^{\infty}\frac{C_kD_mX_k(x)\,Y_m(y)}{K\left(\dfrac{\alpha_k^2}{a^2}+\dfrac{\beta_m^2}{b^2}\right)}\int_{-a}^{a}\int_{-b}^{b} X_k(x)\,Y_m(y)\,Q(x,y,t)\,\mathrm{d}x\,\mathrm{d}y \tag{65}
$$

which, in general, is not well-suited for numerical work. A more suitable expression is readily obtained as

$$
T_{00}(x,y,t)=\sum_k^{\infty}\left(\frac{b}{K}\right)\frac{C_k}{\eta_k}X_k(x)\int_{-a}^{a}\left[\int_0^y \sinh\eta_k\left(\frac{y'}{b}-\frac{y}{b}\right)Q(x,y',t)\,\mathrm{d}y'\right.
$$

$$
\left.+\int_{-b}^{b}\{H(y')\,\Lambda_k(y',y)+H(-y')\,\Lambda_k(y,y')\}\,Q(x,y',t)\,\mathrm{d}y'\right]X_k(x)\,\mathrm{d}x \tag{66}
$$

where

$$
\eta_k=\left(\frac{b}{a}\right)\alpha_k \tag{67}
$$

$$A_k(y, y') = \frac{1}{N(\eta_k)}\left\{\eta_k \cosh \eta_k \left(1 - \frac{y}{b}\right) + B_4 \sinh \eta_k \left(1 - \frac{y}{b}\right)\right\}\left\{\eta_k \cosh \eta_k \left(1 + \frac{y'}{b}\right) + B_3 \sinh \eta_k \left(1 + \frac{y'}{b}\right)\right\}. \quad (68)$$

Alternatively, $T_{00}(x, y, t)$ can also be expressed as

$$T_{00}(x, y, t) = \sum_m^{\infty} \left(\frac{a}{K}\right) \frac{D_m}{\xi_m} Y_m(y) \int_{-b}^{b} \left[\int_0^x \sinh \xi_m \left(\frac{x'}{a} - \frac{x}{a}\right) Q(x', y, t)\, \mathrm{d}x' + \int_{-a}^{a} \{H(x')\, \Omega_m(x', x) + H(-x')\, \Omega_m(x, x')\}\, Q(x', y, t)\, \mathrm{d}x' \right] Y_m(y)\, \mathrm{d}y \quad (69)$$

where

$$\xi_m = \left(\frac{a}{b}\right) \beta_m \quad (70)$$

$$\Omega_m(x, x') = \frac{1}{M(\xi_m)}\left\{ \xi_m \cosh \xi_m \left(1 - \frac{x}{a}\right) + B_2 \sinh \xi_m \left(1 - \frac{x}{a}\right)\right\}\left\{\xi_m \cosh \xi_m \left(1 + \frac{x'}{a}\right) + B_1 \sinh \xi_m \left(1 + \frac{x'}{a}\right)\right\}. \quad (71)$$

Similarly, $U(x, y, z, t)$ and $V(x, y, z, t)$ reduce to

$$U(x, y, t) = \sum_m^{\infty} \left(\frac{a}{K}\right) \frac{D_m}{M(\xi_m)} Y_m(y) \left[\left\{\xi_m \cosh \xi_m \left(1 - \frac{x}{a}\right) + B_2 \sinh \xi_m \left(1 - \frac{x}{a}\right)\right\}\int_{-b}^{b} f_1(y, t)\, Y_m(y)\, \mathrm{d}y + \left\{\xi_m \cosh \xi_m \left(1 + \frac{x}{a}\right) + B_1 \sinh \xi_m \left(1 + \frac{x}{a}\right)\right\}\int_{-b}^{b} f_2(y, t)\, Y_m(y)\, \mathrm{d}y\right] \quad (72)$$

and

$$V(x, y, t) = \sum_k^{\infty} \left(\frac{b}{K}\right) \frac{C_k}{N(\eta_k)} X_k(x)\left[\left\{\eta_k \cosh \eta_k \left(1 - \frac{y}{b}\right) + B_4 \sinh \eta_k \left(1 - \frac{y}{b}\right)\right\}\int_{-a}^{a} f_3(x, t)\, X_k(x)\, \mathrm{d}x + \left\{\eta_k \cosh \eta_k \left(1 + \frac{y}{b}\right) + B_3 \sinh \eta_k \left(1 + \frac{y}{b}\right)\right\}\int_{-a}^{a} f_4(x, t)\, X_k(x)\, \mathrm{d}x\right] \quad (73)$$

respectively. Expression (72) gives the sum of $T_{01}(x, y, t)$ and $T_{02}(x, y, t)$, and expression (73) gives the sum of $T_{03}(x, y, t)$ and $T_{04}(x, y, t)$. The function W is, of course, zero in this case.

(2) *One-dimensional problems*

Another special case of practical importance is the one in which f_1, f_2, F and Q are independent of y and z and, in addition, the faces $y = \pm b$ and $z = \pm c$ are insulated, i.e., $h_3 = h_4 = h_5 = h_6 = f_3 =$

$f_4 = f_5 = f_6 = 0$. This is the case of the one-dimensional unsteady flow of heat in a slab of thickness 2a. Expression (60) then becomes

$$T(x,t) = \sum_{j=0}^{2} T_{0j}(x,t) + \sum_{k=1}^{\infty} C_k X_k(x)\, e^{-\alpha_k^2\left(\frac{\kappa t}{a^2}\right)} \left\{ \int_{-a}^{a} X_k(x)\left[F(x) - \sum_{j=0}^{2} T_{0j}(x,0)\right] dx \right.$$
$$\left. - \sum_{j=0}^{2} \int_0^t e^{\alpha_k^2\left(\frac{\kappa\tau}{a^2}\right)} \int_{-a}^{a} X_k(x) \frac{\partial T_{0j}(x,\tau)}{\partial \tau}\, dx\, d\tau \right\}. \tag{74}$$

Likewise, the expressions alternative to (74) follow from (62) and (63), respectively, as

$$T(x,t) = \sum_{j=0}^{2} T_{0j}(x,t) + \sum_{k=1}^{\infty} C_k X_k(x)\, e^{-\alpha_k^2\left(\frac{\kappa t}{a^2}\right)} \left\{ \int_{-a}^{a} X_k(x)\left[F(x) - \frac{a^2}{K\alpha_k^2} Q(x,0)\right] dx \right.$$
$$\left. - \frac{a^2}{K\alpha_k^2}\left[f_1(0) + X_k(a) f_2(0) + \int_0^t e^{\alpha_k^2\left(\frac{\kappa\tau}{a^2}\right)} \frac{\partial}{\partial \tau}\left\{ \int_{-a}^{a} X_k(x) Q(x,\tau)\, dx + f_1(\tau) + X_k(a) f_2(\tau)\right\} d\tau \right]\right\} \tag{75}$$

and

$$T(x,t) = \sum_{k}^{\infty} C_k X_k(x)\, e^{-\alpha_k^2\left(\frac{\kappa t}{a^2}\right)} \left\{ \int_{-a}^{a} X_k(x) F(x)\, dx \right.$$
$$\left. + \frac{\kappa}{K} \int_0^t e^{\alpha_k^2\left(\frac{\kappa\tau}{a^2}\right)} \left[\int_{-a}^{a} X_k(x) Q(x,\tau)\, dx + f_1(\tau) + X_k(a) f_2(\tau)\right] d\tau \right\}. \tag{76}$$

The functions $T_{0j}(x, t)$ appearing in (74) and (75) readily follow from the corresponding expressions in the two-dimensional case. Thus, (65) or (66) yields

$$T_{00}(x,t) = \frac{a^2}{K} \sum_{k=1}^{\infty} \frac{C_k}{\alpha_k^2} X_k(x)\, Q(x,t)\, dx. \tag{77}$$

This expression, however, is not well-suited for numerical work. On the other hand, expression (69) gives $T_{00}(x, t)$ in closed form as

$$T_{00}(x,t) = \frac{1}{K}\int_0^x (x' - x)\, Q(x',t)\, dx' + \frac{a}{K(B_1 + B_2 + 2B_1B_2)}$$
$$\left\{ \left[1 + B_2\left(1 - \frac{x}{a}\right)\right] \int_{-a}^{0} \left[1 + B_1\left(1 + \frac{x'}{a}\right)\right] Q(x',t)\, dx' \right.$$
$$\left. + \left[1 + B_1\left(1 + \frac{x}{a}\right)\right] \int_0^{a} \left[1 + B_2\left(1 - \frac{x'}{a}\right)\right] Q(x',t)\, dx' \right\}. \tag{78}$$

Finally, $U(x, t)$ follows readily from (72) as

$$U(x, t) = \frac{a}{K(B_1 + B_2 + 2B_1B_2)}\left\{\left[1 + B_2\left(1 - \frac{x}{a}\right)\right]f_1(t) + \left[1 + B_1\left(1 + \frac{x}{a}\right)\right]f_2(t)\right\}. \quad (79)$$

With $f_1(t) = 0$, expression (79) gives $T_{02}(x, t)$ and, with $f_2(t) = 0$, it gives $T_{01}(x, t)$. The functions V and W are, of course, zero in this case.

(3) *The special problem of Cobble*

The problem treated by Cobble [1] is a special case of the two-dimensional problem the solution of which is given in the form expressed by (63). To demonstrate this we let $h_1 = h_2 = h_3 = h_4 = h$ and $f_1 = f_2 = f_3 = f_4 = 0$ in (63). Furthermore, we replace $2a$ by l, $2b$ by a, $(l/2 + x)$ by x, $(a/2 + y)$ by y, and let

$$\alpha_k' = 2\alpha_k, \qquad \beta_k' = 2\beta_k. \quad (80)$$

Equations (12a) and (12b) then give

$$\left.\begin{aligned} B_1 = B_2 &= \frac{1}{2}\left(\frac{hl}{K}\right) = \frac{1}{2}N_{B_i}(l) \\ B_3 = B_4 &= \frac{1}{2}\left(\frac{ha}{K}\right) = \frac{1}{2}N_{B_i}(a) \end{aligned}\right\}. \quad (81)$$

Expressions (10a) and (10b) now read

$$\left.\begin{aligned} X_k(x) &= \cos\alpha_k'\left(\frac{x}{l}\right) + \frac{N_{B_i}(l)}{\alpha_k'}\sin\alpha_k'\left(\frac{x}{l}\right) \\ Y_m(y) &= \cos\beta_m'\left(\frac{y}{a}\right) + \frac{N_{B_i}(a)}{\beta_m'}\sin\beta_m'\left(\frac{y}{a}\right) \end{aligned}\right\} \quad (82)$$

where, in view of (11a) and (11b), α_k' and β_m' are, respectively, the non-negative roots of

$$\tan\alpha_k' = \frac{2\alpha_k' N_{B_i}(l)}{(\alpha_k')^2 - N_{B_i}^2(l)}\,. \quad (83a)$$

and

$$\tan\beta_m' = \frac{2\beta_m' N_{B_i}(a)}{(\beta_m')^2 - N_{B_i}^2(a)} \quad (83b)$$

Equations (13a), (13b) and (61) now become, respectively,

$$\left.\begin{aligned} \frac{1}{C_k} &= \frac{l}{2(\alpha_k')^2}\left[(\alpha_k')^2 + N_{B_i}^2(l) + 2N_{B_i}(l)\right], \\ &= l, \qquad\qquad \alpha_0' = 0 \end{aligned}\right\} \quad (84a)$$

$$\left.\begin{aligned} \frac{1}{D_m} &= \frac{a}{2(\beta_m')^2}\left[(\beta_m')^2 + N_{B_i}^2(a) + 2N_{B_i}(a)\right], \\ &= a, \qquad\qquad \beta_0' = 0 \end{aligned}\right\} \quad (84b)$$

and

$$\lambda_{km}^2 = \left(\frac{\alpha_k'}{l}\right)^2 + \left(\frac{\beta_m'}{a}\right)^2. \tag{85}$$

In view of the substitutions (4) and (9) as applied to the problem under consideration, expression (63) finally leads to

$$T'(x, y, t) = \sum_k^\infty \sum_m^\infty C_k D_m X_k(x)\, Y_m(y)\, e^{-(\lambda_{km}^2 + k)\kappa t} \left\{ \int_0^l \int_0^a X_k(x)\, Y_m(y)\, F(x, y)\, dx\, dy \right.$$
$$\left. + \frac{\kappa}{K} \int_0^t e^{(\lambda_{km}^2 + k)\kappa\tau} \left[\int_0^l \int_0^a X_k(x)\, Y_m(y)\, Q'(x, y, \tau)\, dx\, dy \right] d\tau \right\}. \tag{86}$$

Identifying the quantity

$$\kappa(\lambda_{km}^2 + k) \equiv \kappa\left[\left(\frac{\alpha_k'}{l}\right)^2 + \left(\frac{\beta_m'}{a}\right)^2 + k\right]$$

appearing in (86) with $\psi_{m,n}$ given by equation (39) in [1], and in view of (81), (82), (83) and (84), we have the equivalence of the expression (86) and the one given in [1] by the combination of equations (45), (38), (39), (40), (37), (36), (33), (32) and (22) of [1]. It should be noted that the eigenvalues β_m and β_n appearing in equation (45) of [1] correspond, respectively, to the eigenvalues α_k' and β_m' defined by equations (83a) and (83b), and that the use of the same symbol β in [1] for the two (and, in general, different) sets of eigenvalues α' and β' may lead to ambiguous interpretation of expression (45) of [1] in the course of numerical computation.

Lastly, it is to be noted that the expressions (60) and (62) may also be readily specialized for this problem leading to expressions alternative to (86) and better suited for the treatment of cases where $Q'(x, y, t)$ is a continuous pulse released at $t = 0$.

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Résumé—On présente des expressions générales pour l'échauffement tridimensionnel et instationnaire d'un parallélipipède rectangle plein fini sous l'influence d'une source de chaleur volumique arbitraire et d'une distribution arbitraire de température initiale lorsque l'on impose sur les six faces planes des conditions aux limites du type convectif dépendant du temps. Ces expressions, données sous différentes formes et non disponibles jusqu'à présent, contiennent la solution de nombreux problèmes spéciaux, technologiquement importants. On en déduit les expressions générales correspondantes pour le rectangle et pour la plaque comme cas limites. On montre que le problème particulier traité récemment par Cobble est un cas très spécial des résultats obtenus ici pour le parallélipipède.

Zusammenfassung—Die Untersuchung liefert allgemeine Ausdrücke für das Problem der dreidimensionalen, instationären Aufheizung eines endlichen, rechtwinkeligen Festkörperparallelepipeds unter dem Einfluss einer beliebigen Wärmequellendichteund Anfangstemperaturverteilung, wenn an den sechs ebenen Oberflächen zeitabhängige Randbedingungen dritter Art (Konvektion) vorgeschrieben werden.

Diese, in verschiedener Formulierung vorgelegten, und bisher noch nicht bekannten Ausdrücke, enthalten die Lösung zahlreicher spezieller Probleme von technologischer Bedeutung. Entsprechende allgemeine Ausdrücke für das Rechteckprisma und die ebene Platte werden als Grenzfälle abgeleitet.

Es zeigt sich, dass das kürzlich von Cobble behandelte Sonderproblem einen sehr speziellen Fall der hier für das Parallelepiped abgeleiteten Ergebnisse darstellt.

Аннотация—Приводятся общие выражения для трехмерного, нестационарного нагрева конечного твердого прямоугольного параллелепипеда при произвольном объемном источнике тепла и при произвольном распределении начальной температуры, когда на шести поверхностях задаются конвективные граничные условия, зависящие от времени. Эти выражения представлены в различных видах и ранее не были известны, а также содержат решение различных технологических задач. В качестве предельных случаев выведены соответствующие общие выражения для прямоугольника и плиты. Показано, что недавно рассмотренная Гобблем задача является частным случаем рассмотренных здесь результатов для параллелепипеда.